Written Exam for the M.Sc. in Economics Autumn 2013 (Fall Term)

Financial Econometrics A: Volatility Modelling

Question A:

Solution Question A.1: Follows that $\delta_t = \frac{1}{2}\left(\frac{y_t^2}{\sigma_t^4} - \frac{1}{\sigma_t^2}\right)$. Next,

$$E\left(\delta_t^2|\mathcal{F}_{t-1}\right) = \frac{1}{4}E\left(\left(\frac{y_t^2}{\sigma_t^2} - \frac{1}{\sigma_t^2}\right)^2|\mathcal{F}_{t-1}\right) = \frac{1}{4\sigma_t^4}E\left(\frac{y_t^2}{\sigma_t^2} - 1\right)^2 = \frac{1}{2\sigma_t^4}$$

Hence,

$$\begin{aligned}\sigma_t^2 &= \gamma_1 + \gamma_2\left(y_{t-1}^2 - \sigma_{t-1}^2\right) + \gamma_3\sigma_{t-1}^2 \\ &= \gamma_1 + \gamma_2 y_{t-1}^2 + \left(\gamma_3 - \gamma_2\right)\sigma_{t-1}^2\end{aligned}$$

Solution Question A.2: Know if $\alpha + \beta < 1$ then for the classic GARCH model σ_t^2 is weakly mixing. Hence, as $\alpha + \beta = \gamma_3$ the result holds. For the joint process the results carries over by Meitz and Saikkonen (2008) or by theorem in lecture notes.

Solution Question A.3: With $\gamma_3 = \gamma_2$, then $\partial\sigma_t^2/\partial\gamma_2 = \left(y_{t-1}^2 - \sigma_{t-1}^2\right)$ as $\partial\sigma_t^2/\partial\gamma_2 = \left(y_{t-1}^2 - \sigma_{t-1}^2\right) + \left(\gamma_3 - \gamma_2\right)\partial\sigma_{t-1}^2/\partial\gamma_2$ (note the extra term). Moreover,

$$S_{\gamma_2} = \sum_{t=1}^{T}\left(\frac{y_t^2}{\sigma_t^2} - 1\right)\frac{\left(y_{t-1}^2 - \sigma_{t-1}^2\right)}{\gamma_1 + \gamma_2 y_{t-1}^2}.$$

Hence the CLT for weakly mixing processes applies, provided

$$E\left|\frac{\left(y_{t-1}^2 - \sigma_{t-1}^2\right)}{\gamma_1 + \gamma_2 y_{t-1}^2}\right|^2 < \infty.$$

This is the usual considerations as $\left(\frac{y_{t-1}^2}{\gamma_1 + \gamma_2 y_{t-1}^2}\right)^2 \leq \left(\frac{1}{\gamma_2}\right)^2$ and $\left(\frac{\sigma_{t-1}^2}{\gamma_1 + \gamma_2 y_{t-1}^2}\right)^2 \leq \left(\sigma_{t-1}^2/\gamma_2\right)^2$, with $E\sigma_t^4 < \infty$. Used for part of proof for asymptotic normality of estimators etc.

Solution Question A.4: This is the usual GARCH(1,1) misspecification indication as $\alpha + \beta = \gamma_3$. That there is no ARCH effects lefts needs to be explained etc.

Question B:

Solution Question B.1: Simple application of drift criterion with $\delta(\sigma) = 1 + (\log\sigma^2)^2$ implies that $\sigma_t \in \mathbb{R}_+$ satisfies the assumptions of Theorem 1 in the SV lecture notes. Indeed the joint process is also weakly mixing, either by lecture notes theorem or quoting Meitz and Saikkonen (2008)

Solution Question B.2: $\varepsilon_t = \log|z_t| - \mu$, with z_t Gaussian, and therefore ε_t is not Gaussian. Know that $V(\log|z_t|) = \pi^2/8$ and hence $\sigma_\varepsilon^2 = \pi^2/8$.

The linear Kalman filter would not apply as even in the prediction step, we get

$$X_{t|t-1} = E(\rho(X_{t-1})X_{t-1}|Y_{1:t-1}) \neq cE(X_{t-1}|Y_{1:t-1}) = cX_{t-1|t-1}.$$

In the lecture notes the notation, $X_{t|t-1} = \hat{x}_{t|t-1}$ is used.

Solution Question B.3: As noted the linear KF does not apply - neither does the extended KF seem promising as $\rho(x)$ is not differentiable. Would need some simulation based way, such as the particle filter to simulate the likelihood function. Discussion, and/or summary of particle filter estimation is needed here. Note that an alternative would be GMM but this is not covered in notes and letcures. The ox code piece is the Bootstrap proposal draws (details to be included in answer) is missing `rho*x_1*(x_1<=0)` .

Solution Question B.4: An estimator of the variance given the past can be obtained from the predicted latent proces for log-volatility, $h_{t|t-1} = E(h_t|x_{t-1},...,x_1)$, which could be estimated as $\hat{h}_{t|t-1} = n^{-1}\sum_{i=1}^n h_t^{(i)}$, where $\{h_t^{(i)}\}_{i=1}^n$ denotes the particles from the prediction step, i.e. before resampling. The conditional variance is obtained as $\hat{\sigma}_{t,MLE}^2 = \exp(\hat{h}_{t|t-1})$. This is the predicted volatility. The corresponding filtered volatility is based on the filtered latent process, $h_{t|t-1} = E(h_t|x_t,...,x_1)$, also conditioning on the current observed observation x_t. This is estimated using the weighted particles, $\hat{h}_{t|t} = \sum_{i=1}^n \hat{w}_t^{(i)} h_t^{(i)}$, or after resampling, $\hat{h}_{t|t} = n^{-1}\sum_{i=1}^n \tilde{h}_t^{(i)}$, where $\{\tilde{h}_t^{(i)}\}_{i=1}^n$ is obtained by resampling with replacement from the catagorical distribution given by the possible outcomes and corresponding probabilities, $(x_t^{(1)},...,x_t^{(n)}; w_t^{(1)},...,w_t^{(n)})$.